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On Bogoliubov's model of superfluidity

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Abstract. We point out that the conventional Bogoliubov model contains an attractive effective interaction, putting into question its stability. For positive chemical potentials we show instability, making the model unsuitable for explaining superfluidity from first principles. We present an extended model, yielding rigorously the relevant spectrum for superfluidity.

1. Introduction

In order to explain physical phenomena like superfluidity, one is faced with the usual two-body boson interaction problem and with showing for it the existence of a Bose condensation phase transition. Indeed, the existence of the Bose condensate in liquid ^{4}He has been established in neutron-deep inelastic scattering experiments [1–4] and later confirmed by x-ray measurements [5]. Rigorous results in this field are known for the free boson gas [6] and the imperfect boson gas [7–9] together with refinements of these models [10–13]. Very few rigorous results are known for the general case of boson models with two-body interaction potentials—see e.g. [14–18].

A pragmatic procedure for the description of the properties of superfluids, e.g. the derivation of the experimentally observed spectra, was initiated in Bogoliubov's classical papers [19, 20], where he considered a truncated interaction, giving rise to what will be called the Bogoliubov model. For the reader's convenience and in order to establish notation we shall summarize the main steps of this procedure in section 2. On the basis of perturbation theory and variational estimates (see section 3) one can check that the Bogoliubov model contains an *attractive* effective interaction term. This raises the question of whether the model describes a stable nature. For negative values of the chemical potential, a (finite) upper bound for the pressure is derived in section 3. Based on a variational lower estimate for the pressure, we show in section 4 that the model Hamiltonian is unstable for positive values of the chemical potential. This makes the model unsatisfactory for the purpose of explaining the phenomenon of superfluidity; indeed, positive chemical potential is an essential

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condition in order to give a meaningful account of the right excitation spectrum (i.e. gapless, linear for small momentum and presenting a 'roton' minimum). On the other hand, for negative chemical potential, the variational pressure (which is the output of the pragmatic procedure referred to above) equals the free gas pressure. We conclude that Bogoliubov's model is essentially equivalent, in the grand-canonical ensemble, to the free boson gas. Finally, in section 5 we give the 'minimal extension' of the Bogoliubov model which ensures a reasonable thermodynamics for all chemical potentials and show that the physically relevant results on superfluidity are recovered in the extended model.

2. A brief account of Bogoliubov's approach

The main idea underlying Bogoliubov's theory of superfluidity is that switching on interactions in a condensed boson gas may change drastically the collective excitation spectrum without, however, destroying the Bose condensate. The starting point of the formalism is accordingly a boson gas interacting via a weak two-body potential, the Hamiltonian of which is truncated such that only terms which become important in the presence of the condensate are maintained.

More precisely, the main steps in this approach are the following (we refer to Bogoliubov [19, 20] and e.g. to Hugenholz [21] for more details).

(i) Truncation. Let us consider a system of identical bosons of mass m in a cubic box $\Lambda \subset \mathbb{R}^3$ of volume $V = L^3$, with periodic boundary conditions. If $\phi(x)$ denotes the two-body interaction potential and

$$v(q) = \int_{\mathbb{R}^3} \phi(x) e^{-iqx} d^3x \qquad q \in \mathbb{R}^3 \tag{2.1}$$

then its second-quantized Hamiltonian acting in the Fock space $\mathcal{F}_\Lambda$ can be written as

$$H_\Lambda = \sum_k \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k,k',q} v(q) a_{k+q}^* a_{k'-q}^* a_{k'} a_k \tag{2.2}$$

where all sums run over the set

$$\mathcal{K}_\Lambda = \{k \in \mathbb{R}^3 : k^\alpha = \frac{2\pi}{L} n^\alpha,\ n^\alpha \in \mathbb{Z},\ \alpha = 1, 2, 3\}.$$

Here $\varepsilon_k = |k|^2/2m$ is the kinetic energy, and a_k^*, a_k are the usual boson creation and annihilation operators in the one-particle state $\psi_k(x) = V^{-1/2} e^{ikx}$, $k \in \mathcal{K}_\Lambda$ and $x \in \Lambda$; i.e. $a_k^* \equiv a^*(\psi_k) = \int_\Lambda dx \psi_k(x) a^*(x)$; $a^\#(x)$ are the basic boson operators in the Fock space over $L^2(\mathbb{R}^3)$.

If one expects Bose–Einstein condensation (more precisely, macroscopic occupation of the $k = 0$ mode $\langle a_0^* a_0 \rangle_{H_\Lambda} \sim V$ for the finite-volume Gibbs state $\langle - \rangle_{H_\Lambda}$ defined by (2.2)) then, according to Bogoliubov, the most important terms in (2.2) should be those in which at least two operators a_0^*, a_0 appear. One is thus led to consider the following truncated Hamiltonian:

$$H_\Lambda^B = \sum_k \varepsilon_k a_k^* a_k + \frac{1}{V} v(0) a_0^* a_0 \sum_{k \neq 0} a_k^* a_k + \frac{1}{2V} \sum_{k \neq 0} v(k) [a_0^* a_0 (a_k^* a_k + a_{-k}^* a_{-k}) + a_k^* a_{-k}^* a_0^2 + a_0^{*2} a_{-k} a_k] + \frac{1}{2V} v(0) a_0^{*2} a_0^2 . \tag{2.3}$$

(ii) Bogoliubov approximation. Further simplification of this Hamiltonian can be achieved, according to Bogoliubov, from the observation that $a_0^*/\sqrt{V}$ and $a_0/\sqrt{V}$ almost commute for large V, so that they could be replaced in a macroscopic system by complex numbers $\bar{c}$ and c respectively, to be determined self-consistently. Therefore, in this approximation $H_\Lambda^{\mathrm{B}} - \mu N_\Lambda$, where $N_\Lambda = \sum_k a_k^* a_k$, becomes

$$\mathcal{H}_\Lambda^{\mathrm{B}}(c,\mu) = \sum_{k\neq 0}[\varepsilon_k - \mu + |c|^2 v(0)]a_k^* a_k + \frac{1}{2}\sum_{k\neq 0} v(k)[|c|^2 a_k^* a_k + |c|^2 a_{-k}^* a_{-k} + c^2 a_k^* a_{-k}^* + \bar{c}^2 a_{-k} a_k] - \mu |c|^2 V + \tfrac{1}{2} v(0)|c|^4 V \,. \tag{2.4}$$

(iii) Diagonalization. By performing a gauge transformation $a_k \mapsto a_k \mathrm{e}^{\mathrm{i}\varphi}$ ($\varphi = \arg c$), one can replace c in (2.4) by $|c|$. We shall henceforth use the new variable $x = |c|^2$, having the physical meaning of condensate density. The bilinear Hamiltonian $\mathcal{H}_\Lambda^{\mathrm{B}}(\sqrt{x},\mu)$ can be diagonalized by the well known Bogoliubov canonical transformation $a_k = u_k b_k + v_k b_{-k}^*$, leading to

$$\tilde{\mathcal{H}}_\Lambda^{\mathrm{B}}(\sqrt{x},\mu) = \sum_{k\neq 0} E_k b_k^* b_k + \frac{1}{2}\sum_{k\neq 0}(E_k - f_k) - \mu x V + \tfrac{1}{2} v(0) x^2 V \,. \tag{2.5}$$

Here E_k and f_k are functions of x and μ defined as follows:

$$f_k = \varepsilon_k - \mu + x[v(0) + v(k)] \tag{2.6}$$

$$h_k = x v(k) \tag{2.7}$$

$$E_k = (f_k^2 - h_k^2)^{1/2} \tag{2.8}$$

and the corresponding coefficients of the transformation are

$$u_k^2 = \tfrac{1}{2}(f_k/E_k + 1) \qquad v_k^2 = \tfrac{1}{2}(f_k/E_k - 1) \,. \tag{2.9}$$

Of course, this step makes sense only if $f_k > h_k$ for all $k \neq 0$ such that $v(k) \neq 0$ (otherwise the Bogoliubov transformation is trivial—see (2.9)); this constrains x to belong to the subset $\mathcal{D}_\Lambda(\mu) = \{x \geqslant 0 : v(0)x > \mu - \min_{k\neq 0} \varepsilon_k\}$.

(iv) Interpretation. Finally one has to determine x and μ from conditions like, for example, minimum ground-state energy for (2.5) and prescribed particle density ρ. This is done approximately by supposing, again by Bogoliubov's proposal, that all particles are condensed at zero temperature:

$$x = \rho \,. \tag{2.10}$$

On the other hand, from first-order calculations of the ground-state energy [21], the *subsidiary* condition yields

$$\mu = v(0)\rho \,. \tag{2.11}$$

With these values one has

$$E_{k=0} = 0 \quad \text{and} \quad E_k \sim [\rho v(0)/m]^{1/2}|k| \qquad \text{for } k \to 0 \,. \tag{2.12}$$

This structure of the collective excitation spectron explains, according to Landau's criterion [22], the superfluid properties of the system.

3. The stability of H_Λ^{B} and an upper bound for the pressure

We shall work under the following assumptions on the interaction potential ϕ, expressed in terms of its Fourier transform, (2.1):

(ϕ)$v(q)$ is a real continuous function with bounded support, satisfying $v(0) > 0$ and $0 \leqslant v(q) = v(-q) \leqslant v(0)$ for all $q \in \mathbb{R}^3$.

Under these (in fact, even under weaker) conditions it is known [23] that ϕ is superstable and hence that the grand-canonical partition function associated with the full Hamiltonian (2.2)

$$\Xi_\Lambda(\beta, \mu) = \mathrm{Tr}\, \exp[-\beta(H_\Lambda - \mu N_\Lambda)]$$

is finite for all real μ and all $\beta > 0$.

As far as we are aware, the analogous study of the truncated Hamiltonian H_Λ^{B} given by (2.3) has not been undertaken. A preliminary step in this direction is the following rough estimate.

Proposition 3.1. Suppose (ϕ). Then, for every $\mu < 0$ and Λ sufficiently large, $H_\Lambda^{\mathrm{B}} - \mu N_\Lambda$ is bounded from below the self-adjoint operator in $\mathcal{F}_\Lambda$ and the following inequality holds:

$$H_\Lambda^{\mathrm{B}} - \mu N_\Lambda \geqslant \sum_k \left(\varepsilon_k - \mu - \frac{v(0)}{2V}\right) n_k - \frac{1}{2}\phi(0) n_0 + \frac{1}{2} v(0) \frac{n_0^2}{V} \tag{3.1}$$

where $n_k = a_k^* a_k$ are occupation-number operators for modes $k \in \mathcal{K}_\Lambda$.

Proof. Using $[a_k^\#, a_0^\#] = 0$ for $k \neq 0$, one has

$$\begin{aligned} &a_0^* a_0 (a_k^* a_k + a_{-k}^* a_{-k}) + a_k^* a_{-k}^* a_0^2 + a_0^{*2} a_{-k} a_k \\ &\quad = (a_0^* a_k + a_{-k}^* a_0)^* (a_0^* a_k + a_{-k}^* a_0) - a_k^* a_k - a_0^* a_0 \,. \end{aligned} \tag{3.2}$$

Hence, by regrouping terms in (2.3)

$$\begin{aligned} H_\Lambda^{\mathrm{B}} - \mu N_\Lambda = &\sum_{k \neq 0} \left(\varepsilon_k - \mu - \frac{1}{2V} v(k)\right) n_k \\ &+ \frac{1}{2V} \sum_{k \neq 0} v(k) (a_0^* a_k + a_{-k}^* a_0)^* (a_0^* a_k + a_{-k}^* a_0) + \frac{1}{V} v(0) n_0 \sum_{k \neq 0} n_k \\ &+ \frac{1}{2V} v(0) n_0^2 - \left(\mu + \frac{1}{2V} \sum_k v(k)\right) n_0 \,. \end{aligned} \tag{3.3}$$

For $\mu < 0$ and Λ such that $\min_{k \in \mathcal{K}_\Lambda} [\varepsilon_k - \mu - \frac{1}{2V} v(k)] > 0$, all terms but the last in this expression are positive self-adjoint operators with forms having a dense intersection of domains, while the last is a small form-perturbation, from which the first part of the proposition follows [24]. The simple lower bound in (3.1) is obtained by discarding the second and third terms in (3.3), which are positive, and using the assumption $v(q) \leqslant v(0)$. □

An immediate consequence is the following upper bound for the pressure.

Corollary 3.2. For $\mu < 0$,

$$p^{\mathrm{B}}(\beta,\mu) := \lim_{\Lambda\to\infty} \frac{1}{\beta V} \ln \ \mathrm{Tr}\exp[-\beta(H_\Lambda^{\mathrm{B}} - \mu N_\Lambda)] \leqslant p_0(\beta,\mu) + \lim_{\Lambda\to\infty} \frac{1}{\beta V} \ln \left\{ \sum_{n=0}^{\infty} \exp\beta\left[\left(\mu + \frac{1}{2}\phi(0)\right)n - \frac{1}{2}v(0)\frac{n^2}{V}\right]\right\} \qquad (3.4)$$

where $p_0(\beta,\mu)$ is the free gas pressure,

$$p_0(\beta,\mu) = \beta^{-1}\int_{\mathbb{R}^3} \ln(1 - \mathrm{e}^{-\beta(\varepsilon_k - \mu)})^{-1} \mathrm{d}^3k/(2\pi)^{3/2}\,. \qquad (3.5)$$

The last term on the right-hand side of (3.4) vanishes for $\mu < -\frac{1}{2}\phi(0)$ so, for such μ

$$p^{\mathrm{B}}(\beta,\mu) \leqslant p_0(\beta,\mu)\,. \qquad (3.6)$$

To relax the upper bound on the chemical potential in (3.6) we represent the operator (3.3) in the form $\mathcal{H}_\Lambda^{(0)}(\mu) + W_\Lambda$, where

$$\mathcal{H}_\Lambda^{(0)}(\mu) = \sum_{k\neq 0}\left(\varepsilon_k - \mu - \frac{1}{2V}v(k)\right)n_k + \frac{1}{V}v(0)n_0\sum_{k\neq 0}n_k + \frac{1}{2V}v(0)n_0^2 - \left(\mu + \frac{1}{2V}\sum_k v(k)\right)n_0 \qquad (3.7)$$

$$W_\Lambda = \frac{1}{2V}\sum_{k\neq 0} v(k)(a_0^* a_k + a_{-k}^* a_0)^*(a_0^* a_k + a_{-k}^* a_0) \geqslant 0\,.$$

Let $p_\Lambda[H_\Lambda^{\mathrm{B}} - \mu N]$ and $p_\Lambda[\mathcal{H}_\Lambda^{(0)}(\mu)]$ be the corresponding pressures in the finite volume. Then by the Bogoliubov inequality [25] and (3.7) one gets

$$0 \leqslant \frac{1}{V}\langle W_\Lambda\rangle_{H_\Lambda^{\mathrm{B}} - \mu N_\Lambda} \leqslant p_\Lambda[\mathcal{H}_\Lambda^{(0)}(\mu)] - p_\Lambda[H_\Lambda^{\mathrm{B}} - \mu N_\Lambda] \leqslant \frac{1}{V}\langle W_\Lambda\rangle_{\mathcal{H}_\Lambda^{(0)}(\mu)} \qquad (3.8)$$

where $\langle - \rangle_{H_\Lambda^{\mathrm{B}} - \mu N_\Lambda}$ and $\langle - \rangle_{\mathcal{H}_\Lambda^{(0)}(\mu)}$ are corresponding finite-volume Gibbs states. Note that the operator W_Λ (see (3.7)) has non-diagonal and diagonal parts

$$W_\Lambda = \frac{1}{2V}\sum_{k\neq 0} v(k)(a_k^* a_{-k}^* a_0 a_0 + a_0^* a_0^* a_{-k} a_k) + \frac{1}{2V}\sum_{k\neq 0} v(k)[a_k^* a_k(a_0^* a_0 + 1) + a_0^* a_0(a_{-k}^* a_{-k} + 1)] \equiv U_\Lambda^{\mathrm{B}} + D_\Lambda\,. \qquad (3.9)$$

Therefore, for the non-diagonal part we have $\langle U_\Lambda^{\mathrm{B}}\rangle_{\mathcal{H}_\Lambda^{(0)}} = 0$ and consequently by (3.8) one gets

$$0 \leqslant p_\Lambda[\mathcal{H}_\Lambda^{(0)}(\mu)] - p_\Lambda[H_\Lambda^{\mathrm{B}} - \mu N_\Lambda] \leqslant \frac{1}{V}\langle D_\Lambda\rangle_{\mathcal{H}_\Lambda^{(0)}(\mu)}\,. \qquad (3.10)$$

Using an explicit expression (3.7) for $\mathcal{H}_\Lambda^{(0)}(\mu)$ one gets

$$\operatorname{Tr}\exp\{\beta V p_\Lambda[\mathcal{H}_\Lambda^{(0)}(\mu)]\} = \sum_{n_0=0}^{\infty} \exp\left\{-\beta V\left[\frac{v(0)}{2}\left(\frac{n_0}{V}\right)^2 - \left(\mu+\frac{1}{2}\phi(0)\right)\frac{n_0}{V}\right]\right\}$$
$$\times \sum_{\{n_k=0,1,\ldots\}_k} \prod_{k\neq 0} \exp\left[-\beta\left(\varepsilon_k - \mu - \frac{v(k)}{2V} + \frac{v(0)}{V}n_0\right)n_k\right]. \quad (3.11)$$

Then in the thermodynamic limit we obtain

$$p^{(0)}(\beta,\mu) = \lim_\Lambda p_\Lambda[\mathcal{H}_\Lambda^{(0)}(\mu)]$$
$$= \sup_{\rho_0\geqslant 0}\left[-\frac{v(0)}{2}\rho_0^2 + \left(\mu+\frac{1}{2}\phi(0)\right)\rho_0 + p_0(\beta,\mu-v(0)\rho_0)\right]. \quad (3.12)$$

If $[\mu+\frac{1}{2}\phi(0)]\leqslant 0$, then sup in (3.12) is attained at $\overline{\rho}_0 = 0$. So, $p^{(0)}(\beta,\mu) = p_0(\beta,\mu)$ and by (3.10) we again get the estimate (3.6). The same result persists for the weaker condition

$$\mu + \tfrac{1}{2}\phi(0) \leqslant \inf_{\rho_0\geqslant 0}[v(0)\rho_0 + v(\omega)\partial_\mu p_0(\beta,\mu-v(0)\rho_0)]. \quad (3.13)$$

To get the estimate of the pressure $p^{\mathrm{B}}(\beta,\mu)$ from below we again use the Bogoliubov inequality

$$\frac{1}{V}\langle H_\Lambda^{\mathrm{B}} - T_\Lambda\rangle_{H_\Lambda^{\mathrm{B}}-\mu N_\Lambda} \leqslant p_\Lambda[T_\Lambda - \mu N_\Lambda] - p_\Lambda[H_\Lambda^{\mathrm{B}} - \mu N_\Lambda] \leqslant \frac{1}{V}\langle H_\Lambda^{\mathrm{B}} - T_\Lambda\rangle_{T_\Lambda-\mu N_\Lambda} \quad (3.14)$$

where $T_\Lambda = \sum_k \varepsilon_k a_k^* a_k$ is the Hamiltonian of the ideal boson gas. The expectation on the right-hand side of (3.14) can be easily calculated explicitly:

$$\langle H_\Lambda^{\mathrm{B}} - T_\Lambda\rangle_{T_\Lambda-\mu N_\Lambda} = \frac{1}{V}\langle n_0\rangle_{T_\Lambda-\mu N_\Lambda} v(0)\sum_{k\neq 0}\langle n_k\rangle_{T_\Lambda-\mu N_\Lambda}$$
$$+ \frac{1}{2V}\langle n_0\rangle_{T_\Lambda-\mu N_\Lambda}\sum_{k\neq 0} v(k)(\langle n_k\rangle_{T_\Lambda-\mu N_\Lambda} + \langle n_k\rangle_{T_\Lambda-\mu N_\Lambda}). \quad (3.15)$$

For $\mu < 0$ there is no Bose-condensation in the ideal boson gas, i.e.

$$\lim_V \frac{1}{V}\langle n_0\rangle_{T_\Lambda-\mu N_\Lambda} = 0 \qquad (\mu < 0). \quad (3.16)$$

Consequently, by (3.14) and (3.16) we get for $\mu < 0$

$$\lim_V \frac{1}{V}\langle H_\Lambda^{\mathrm{B}} - T_\Lambda\rangle_{T_\Lambda-\mu N_\Lambda} = 0. \quad (3.17)$$

Hence, (3.14) and (3.17) give the following estimate:

$$p_0(\beta,\mu) \leqslant p^{\mathrm{B}}(\beta,\mu) \qquad \mu < 0. \quad (3.18)$$

The inequality (3.18) says that the *interaction* in the Bogoliubov's Hamiltonian (2.8) is *attractive*. This is the reason of the instability of $H_\Lambda^{\mathrm{B}} - \mu N_\Lambda$ (3.3) which we shall discuss in the next section.

Combining (3.6) and (3.18) we get

$$p^{\mathrm{B}}(\beta,\mu) = p_0(\beta,\mu) \quad (3.19)$$

for negative chemical potentials satisfying the condition (3.13).

4. The Bogoliubov approximation for H_Λ^{B} and the variational pressure

The boson Fock space $\mathcal{F}_\Lambda$ can be written as a tensor product $\mathcal{F}_{0\Lambda} \otimes \mathcal{F}'_\Lambda$, where $\mathcal{F}'_\Lambda$ is the Fock space over the orthogonal complement of the constant functions in $L^2(\Lambda)$ and $\mathcal{F}_{0\Lambda}$ is the Fock space over the one-dimensional subspace of constant functions ($k = 0$ mode). For every $c \in \mathbb{C}$ let us consider the coherent vector in $\mathcal{F}_{0\Lambda}$:

$$\psi_0(c) = \mathrm{e}^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} (V^{1/2} c)^k (a_0^*)^k \Omega_0 \tag{4.1}$$

where Ω_0 is the vacuum of $\mathcal{F}_{0\Lambda}$.

As has been clarified by Ginibre [14], the Bogoliubov approximation to a Hamiltonian H in $\mathcal{F}_\Lambda$ is the operator $H(c)$ on $\mathcal{F}'_\Lambda$ defined by its quadratic form

$$(\psi'_1, H(c)\psi'_2) = (\psi_0(c) \otimes \psi'_1, H \psi_0(c) \otimes \psi'_2) \tag{4.2}$$

for all $\psi'_1, \psi'_2 \in \mathcal{F}'_\Lambda$ for which $\psi_0(c) \otimes \psi'_{1,2}$ are in the form domain of H. The advantage of this formulation of Bogoliubov's approximation is that it provides a variational principle for the pressure allowing one to determine the value of c.

Proposition 4.1. [14] Let H be a self-adjoint operator in $\mathcal{F}_\Lambda$ such that $\exp(-\beta H)$ has finite trace for $\beta > 0$ and suppose that $H(c)$ defined by (4.2) is self-adjoint in $\mathcal{F}'_\Lambda$ for every $c \in \mathbb{C}$. There, $\exp[-\beta H(c)]$ has finite trace in $\mathcal{F}'_\Lambda$ for all $c \in \mathbb{C}$ and $\beta > 0$, and

$$\mathrm{Tr}_{\mathcal{F}'_\Lambda} \exp[-\beta H(c)] \leqslant \mathrm{Tr}_{\mathcal{F}_\Lambda} \exp(-\beta H) . \tag{4.3}$$

Proof. Using a product basis in $\mathcal{F}_\Lambda$ in calculating $\mathrm{Tr}_{\mathcal{F}_\Lambda}(-) = \mathrm{Tr}_{\mathcal{F}_{0\Lambda} \otimes \mathcal{F}'_\Lambda}(-)$, one gets

$$\mathrm{Tr}_{\mathcal{F}_\Lambda} \exp(-\beta H) \geqslant \sum_n [\psi_0(c) \otimes \psi'_n, \exp(-\beta H) \psi_0(c) \otimes \psi'_n]$$

where $\{\psi'_n\}_n$ is an arbitrary orthonormal basis in $\mathcal{F}'_\Lambda$. Now, whenever $\psi_0(c) \otimes \psi'_n$ are in the form domain of H, Peierl's inequality [23] gives, by the definition (4.2) of $H(c)$,

$$(\psi_0(c) \otimes \psi'_n, \exp(-\beta H) \psi_0(c) \otimes \psi'_n) \geqslant \exp[-\beta(\psi'_n, H(c)\psi'_n)] .$$

Therefore

$$\mathrm{Tr}_{\mathcal{F}_\Lambda} \exp(-\beta H) \geqslant \sup_{\{\psi'_n\}_n} \sum_n \exp[-\beta(\psi'_n, H(c)\psi'_n)] \tag{4.4}$$

where the sup is over all orthonormal bases of $\mathcal{F}'_\Lambda$ contained in the form domain of $H(c)$. The finiteness of the right-hand side of (4.4) implies that $\exp[-\beta H(c)]$ is trace-class in $\mathcal{F}'_\Lambda$ and

$$\mathrm{Tr}_{\mathcal{F}'_\Lambda} \exp[-\beta H(c)] = \sup_{\{\psi'_n\}_n} \sum_n \exp[-\beta(\psi'_n, H(c)\psi'_n)] .$$

□

In the case $H = H_\Lambda^{\mathrm{B}} - \mu N_\Lambda$, $H(c)$ defined in (4.2) will be $\mathcal{H}_\Lambda^{\mathrm{B}}(c,\mu)$ (equation (2.4)). Denoting

$$\tilde{p}_\Lambda^{\mathrm{B}}(\beta,\mu;c) = \frac{1}{\beta V}\log \mathrm{Tr}_{\mathcal{F}'_\Lambda} \exp[-\beta \mathcal{H}_\Lambda^{\mathrm{B}}(c,\mu)] \tag{4.5}$$

one obtains the following.

Corollary 4.2. If $p_\Lambda(\beta,\mu) < \infty$ for some $\beta > 0$ and $\mu \in \mathbb{R}$, then $\tilde{p}_\Lambda^{\mathrm{B}}(\beta,\mu;c) < \infty$ for all $c \in \mathbb{C}$ and

$$p_\Lambda^{\mathrm{B}}(\beta,\mu) \geqslant \sup_{c\in\mathbb{C}} \tilde{p}_\Lambda^{\mathrm{B}}(\beta,\mu;c). \tag{4.6}$$

Otherwise stated, if for some $\beta > 0$ and $\mu \in \mathbb{R}$ the supremum in (4.6) is infinite, then $\exp[-\beta(H_\Lambda^{\mathrm{B}} - \mu N_\Lambda)]$ will not have finite trace and therefore the finite-volume pressure $p_\Lambda^{\mathrm{B}}(\beta,\mu)$ will not be defined ($= +\infty$).

We are left with the task of calculating $\tilde{p}_\Lambda^{\mathrm{B}}(\beta,\mu;c)$ and its supremum. As already mentioned in section 2, if $x = |c|^2 \in \mathcal{D}_\Lambda(\mu)$, $\mathcal{H}_\Lambda^{\mathrm{B}}(\sqrt{x},\mu)$ can be diagonalized by a canonical transformation, which is formally a tensor product of unitaries $\mathcal{U}_k$, each operating in the Fock space corresponding to a pairs $\{k,-k\}_{k\in\mathcal{K}_\Lambda}$, $k \neq 0$ for which $v(k) \neq 0$. Note that $\mathcal{U}_k = \mathbb{1}_k$ for $k \notin \operatorname{supp} v(\cdots)$—see (2.9). Under our assumption (ϕ), for every finite Λ there will be a *finite number* of $k \in \mathcal{K}_\Lambda$ for which $v(k) \neq 0$, so the above tensor product is well defined as a unitary in $\mathcal{F}'_\Lambda$. We can therefore use (2.5) for all Λ; $\beta > 0$, $\mu \in \mathbb{R}$ and $x \in \mathcal{D}_\Lambda(\mu)$ to calculate $\tilde{p}_\Lambda^{\mathrm{B}}(\beta,\mu;\sqrt{x})$:

$$\tilde{p}_\Lambda^{\mathrm{B}}(\beta,\mu;\sqrt{x}) = \frac{1}{\beta V}\sum_{k\neq 0}\ln(1-\mathrm{e}^{-\beta E_k})^{-1} - \frac{1}{2V}\sum_{k\neq 0}(E_k - f_k) + \mu x - \tfrac{1}{2}v(0)x^2 \tag{4.7}$$

where E_k, f_k are given by (2.6)–(2.8).

Proposition 4.3. Under the assumption (ϕ), for every $\mu > 0$ and all sufficiently large Λ, $\tilde{p}_\Lambda^{\mathrm{B}}(\beta,\mu;c)$ is not bounded above, therefore $\exp[-\beta(H_\Lambda^{\mathrm{B}} - \mu N_\Lambda)]$ does not have finite trace.

Proof. Let $\mu > 0$. Then, for Λ sufficiently large

$$\varepsilon_-(\Lambda) = \min_{0\neq k\in\mathcal{K}_\Lambda} \varepsilon_k < \mu \tag{4.8}$$

and so $\mathcal{D}_\Lambda(\mu)$ is given by

$$x > [\mu - \varepsilon_-(\Lambda)]/v(0). \tag{4.9}$$

For $x \in \mathcal{D}_\Lambda(\mu)$ we use (4.7). As for $x \searrow [\mu - \varepsilon_-(\Lambda)]/v(0)$, $\min_{0\neq k\in\mathcal{K}_\Lambda} E_k \searrow 0$, it follows that $\tilde{p}_\Lambda^{\mathrm{B}}(\beta,\mu;\sqrt{x}) \to \infty$ due to the logarithmic terms in (4.7), corresponding to $\varepsilon_k = \varepsilon_-(\Lambda)$, which diverge in this limit. $\square$

Proposition 4.4. Suppose (ϕ) and, moreover,

$$v(0) > \frac{1}{2(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathrm{d}^3k \; v(k)^2/\varepsilon_k . \tag{4.10}$$

Then, for all $\beta > 0$, $\mu < 0$ and for all sufficiently large Λ, the supremum of $\tilde{p}^{\mathrm{B}}_\Lambda(\beta, \mu; \sqrt{x})$ is attained at $\bar{x} = 0$. Therefore in the thermodynamic limit

$$p^{\mathrm{B}}(\beta, \mu) \geqslant p_0(\beta, \mu) \qquad \beta > 0, \; \mu < 0 . \tag{4.11}$$

Proof. If $\mu < 0$, $\mathcal{D}_\Lambda(\mu) = [0, \infty)$ and $\tilde{p}^{\mathrm{B}}_\Lambda(\beta, \mu; \sqrt{x})$, (4.7) will be differentiable with respect to x on $(0, \infty)$, and will go to $-\infty$ for $x \to \infty$. Therefore its supremum is attained either at $x = 0$, or at a positive solution of the equation

$$\frac{\partial \tilde{p}^{\mathrm{B}}_\Lambda}{\partial x} \equiv -\frac{1}{V} \sum_{k \neq 0} (1 - \mathrm{e}^{-\beta E_k})^{-1} \frac{\partial E_k}{\partial x} - \frac{1}{2V} \sum_{k \neq 0} \left(\frac{\partial E_k}{\partial x} - \frac{\partial f_k}{\partial x} \right) + \mu - x v(0) = 0 . \tag{4.12}$$

Using (2.6)–(2.8) one calculates

$$\frac{\partial f_k}{\partial x} = v(0) + v(k) \qquad \frac{\partial E_k}{\partial x} = E_k^{-1} [f_k v(0) + (f_k - h_k) v(k)] . \tag{4.13}$$

Clearly, if $\mu < 0$, $\partial E_k / \partial x > 0$ on $(0, \infty)$ for all $k \in \mathcal{K}_\Lambda$. Therefore

$$\frac{\partial \tilde{p}^{\mathrm{B}}_\Lambda}{\partial x} < \left. \frac{\partial \tilde{p}^{\mathrm{B}}_\Lambda}{\partial x} \right|_{\beta = \infty} = -\frac{1}{2V} \sum_{k \neq 0} \left(\frac{\partial E_k}{\partial x} - \frac{\partial f_k}{\partial x} \right) + \mu - x v(0) . \tag{4.14}$$

Further, for fixed $x > 0$, $\left. \frac{\partial \tilde{p}^{\mathrm{B}}_\Lambda}{\partial x} \right|_{\beta = \infty}$ is an increasing function of μ, because its derivative

$$\frac{\partial}{\partial \mu} \left(\left. \frac{\partial \tilde{p}^{\mathrm{B}}_\Lambda}{\partial x} \right|_{\beta = \infty} \right) = -\frac{1}{2V} \sum_{k \neq 0} \frac{\partial^2 E_k}{\partial \mu \partial x} + 1 = \frac{1}{2V} \sum_{k \neq 0} \frac{(\varepsilon_k - \mu) x v(k)^2}{E_k^3} + 1$$

is manifestly positive for $\mu < 0$. Thus

$$\frac{\partial \tilde{p}^{\mathrm{B}}_\Lambda}{\partial x}(\infty, \mu; x) < \frac{\partial \tilde{p}^{\mathrm{B}}_\Lambda}{\partial x}(\infty, 0; x) \equiv f_\Lambda(x) . \tag{4.15}$$

$f_\Lambda(x)$ is a concave function on $(0, \infty)$, because

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f_\Lambda(x) = -\frac{1}{2V} \sum_{k \neq 0} \left. \frac{\partial^3 E_k}{\partial x^3} \right|_{\mu = 0} = -\frac{1}{2V} \sum_{k \neq 0} \frac{3 v(k)^2 \varepsilon_k^2}{E_k^4} \left. \frac{\partial E_k}{\partial x} \right|_{\mu = 0} < 0 . \tag{4.16}$$

Finally, $f_\Lambda(0) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}x} f_\Lambda(0) = -v(0) + \frac{1}{2V} \sum_{k \neq 0} \frac{v(k)^2}{\varepsilon_k}$$

is negative for sufficiently large Λ if condition (4.10) is fulfilled. Therefore

$$f_\Lambda(x) < 0 \qquad \text{for } x \in (0, \infty) . \tag{4.17}$$

Combining (4.14), (4.15) and (4.16), we conclude that (4.12) has no positive solution, wherefrom

$$\max_{x \in [0, \infty)} \tilde{p}^{\mathrm{B}}_\Lambda(\beta, \mu; \sqrt{x}) = \tilde{p}^{\mathrm{B}}_\Lambda(\beta, \mu; 0) = \frac{1}{\beta V} \sum_{k \neq 0} \ln(1 - \mathrm{e}^{-\beta(\varepsilon_k - \mu)})^{-1} .$$

□

Remarks.

(i) The condition (4.10) is a weakness condition on the potential: if $v(k) = g \cdot \varphi(k)$ with φ some fixed function satisfying (ϕ), (4.10) holds for g sufficiently small.

(ii) It is known [14] that Bogoliubov's approximation is exact when operated on the full Hamiltonian H_Λ (equation (2.2)), under condition (ϕ), i.e. (4.6) becomes an equality in the thermodynamic limit. (3.6) and (4.11) prove the exactness of Bogoliubov's approximation $\mathcal{H}_\Lambda^{\mathrm{B}}(\sqrt{x}, \mu)|_{x=\bar{x}}$ to $H_\Lambda^{\mathrm{B}} - \mu N_\Lambda$ for $\mu < -\frac{1}{2}\phi(0)$. We conjecture that $p^{\mathrm{B}}(\beta, \mu) = p_0(\beta, \mu)$ for all $\mu < 0$. If that were true, and having in view proposition 4.3 too, it would follow that Bogoliubov's Hamiltonian H_Λ^{B} predicts the same thermodynamics as the free Hamiltonian.

5. A model for superfluidity

As shown in the previous sections the Bogoliubov truncation is not able to provide rigorously an explanation of the phenomenon of superfluidity.

Here we examine a possible way out of this situation. The idea, which in fact has already been put forward before [26], is to make a milder truncation of the full Hamiltonian (1.2). We propose here a 'minimal' extension of the Bogoliubov Hamiltonian (2.3) keeping, in addition to $\frac{1}{2V} v(0) \sum_k a_0^* a_k^* a_k a_0$, all other 'forward scattering' terms

$$\frac{1}{2V} v(0) \sum_{k,k' \neq 0} a_k^* a_{k'}^* a_{k'} a_k \, .$$

This term will ensure the superstability of the truncated Hamiltonian. Therefore we consider the model

$$\tilde{H}_\Lambda^{\mathrm{B}} = \sum_k \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k \neq 0} v(k) (a_k^* a_{-k}^* a_0^2 + a_0^{*2} a_{-k} a_k) + \frac{1}{V} a_0^* a_0 \sum_{k \neq 0} v(k) a_k^* a_k + \frac{v(0)}{2V} N_\Lambda (N_\Lambda - 1) \, . \tag{5.1}$$

The Hamiltonian (5.1) resembles the one exploited in the Bogoliubov theory [20–22] after approximation $n_0 \simeq N_\Lambda$. In our model (5.1), (i) we keep a_0, a_0^*-operators; (ii) we treat the $N_\Lambda(N_\Lambda - 1)$-term as operator in $\mathcal{F}_\Lambda$.

The last term in (5.1) is recognized to be the interaction term of the so-called imperfect Bose gas model [27], which has been extensively studied in a rigorous manner [8–10, 28].

The difficulty with the model $\tilde{H}_\Lambda^{\mathrm{B}}$ (5.1) in contradistinction to H_Λ^{B}, is that the result of the Bogoliubov approximation (2.2) no longer yields a bilinear Hamiltonian. This point is treated in an approximate way in [20–22, 26], with the assumption that the total particle number N_Λ is fixed, i.e. keeping the N_Λ^2-term unchanged while performing the Bogoliubov transformation in order to diagonalize the bilinear terms. It is clear that the Bogoliubov transformation does not preserve the particle number. In what follows we develop rigorous arguments *to remedy this discrepancy* or to prove the essential ingredients of superfluidity.

First we show that our model (5.1) has good thermodynamic behaviour in the sense that the Hamiltonian is *superstable* [23].

Proposition 5.1. For each finite volume Λ, one has

$$\tilde{H}_\Lambda^{\mathrm{B}} \geqslant -\left(\frac{1}{2}\phi(0) + \frac{v(0)}{V}\right) N_\Lambda + \frac{v(0)}{2V} N_\Lambda^2 . \tag{5.2}$$

Proof. Use again the operator equality (3.2). Then one gets

$$(a_k^* a_{-k}^* a_0^2 + a_0^{*2} a_{-k} a_k) + a_0^* a_0 (a_k^* a_k + a_{-k}^* a_{-k}) \geqslant -a_k^* a_k - a_0^* a_0 .$$

Therefore, Hamiltonian (5.1) has the following estimate from below:

$$\begin{aligned}\tilde{H}_\Lambda^{\mathrm{B}} &\geqslant \sum_k \epsilon_k n_k - \frac{1}{2V}\sum_{k\neq 0} v(k)(n_k + n_0) + \frac{v(0)}{2V} N_\Lambda (N_\Lambda - 1) \\ &\geqslant -\left(\frac{1}{2}\phi(0) N_\Lambda + \frac{v(0)}{V} N_\Lambda\right) + \frac{v(0)}{2V} N_\Lambda^2 .\end{aligned}$$

As $0 \leqslant v(k) \leqslant v(0)$ one gets the lower bound (5.2), proving the superstability of the model. □

Now we perform a Bogoliubov approximation (4.2) in the *second* and *third* terms of $\tilde{H}_\Lambda^{\mathrm{B}}$ (5.1) and we get the following refinement of the Bogoliubov model:

$$\begin{aligned}\tilde{H}_{\Lambda,c}^{\mathrm{B}} = &\sum_k \varepsilon_k a_k^* a_k + \frac{1}{2}\sum_{k\neq 0} v(k)(a_k^* a_{-k}^* c^2 + \bar{c}^2 a_{-k} a_k) \\ &+ |c|^2 \sum_{k\neq 0} v(k) a_k^* a_k + \frac{v(0)}{2V} N_\Lambda (N_\Lambda - 1)\end{aligned} \tag{5.3}$$

with the implicit assumption that $|c|^2$ represents the condensate density in the zero mode. Clearly model (5.3) remains superstable.

Remark that our model $\tilde{H}_{\Lambda,c}^{\mathrm{B}}$ (5.3) does not coincide with the Bogoliubov approximation $\tilde{H}_\Lambda^{\mathrm{B}}(c)$ which we should obtain following the procedure of section 4. The Hamiltonian (5.3) is still acting on the boson Fock space $\mathcal{F}_\Lambda$. Later we come back to the comparison between our model $\tilde{H}_{\Lambda,c}^{\mathrm{B}}$ (5.3) and the full Bogoliubov approximation $\tilde{H}_\Lambda^{\mathrm{B}}(c)$ of the model $\tilde{H}_\Lambda^{\mathrm{B}}$ (5.1).

In this work we are interested in the spectrum of the Hamiltonian $\tilde{H}_{\Lambda,c}^{\mathrm{B}}$ (5.3) in an equilibrium state at inverse temperature $\beta = 1/kT$ and chemical potential μ, already in the thermodynamic limit.

There are many ways to characterize such equilibrium states, e.g. as limit Gibbs states with special boundary conditions, or as states minimizing the free energy functional, etc. We will characterize them as the states satisfying the energy–entropy balance correlation inequalities [29]. This characterization will be particularly useful for our purposes. Therefore we define the equilibrium state denoted by $\langle\cdots\rangle_{\beta,\mu}$, as a state satisfying for all local observables X (in the domain of the commutator with $\tilde{H}_{\Lambda,c}^{\mathrm{B}}$) the inequality

$$\lim_\Lambda \beta \langle X^* [\tilde{H}_{\Lambda,c}^{\mathrm{B}} - \mu N_\Lambda, X]\rangle_{\beta,\mu} \geqslant \langle X^* X\rangle_{\beta,\mu} \ln \frac{\langle X^* X\rangle_{\beta,\mu}}{\langle X X^*\rangle_{\beta,\mu}} \tag{5.4}$$

supplemented with the condensate equation

$$c = \lim_{\Lambda} \frac{1}{\sqrt{V}} \langle a_0 \rangle_{\beta,\mu} \tag{5.5}$$

consistent with the Bogoliubov approximation (4.2).

Condition (5.5) is in agreement with the usual interpretation of $|c|^2$ being the condensate density (see section 4).

Now we are interested in the solutions of (5.4), (5.5).

Before going on we want to indicate already that the characterization (5.4) makes clear that the equilibrium state $\langle \cdots \rangle_{\beta,\mu}$ is not determined by the Hamiltonian $\tilde{H}^{\mathrm{B}}_{\Lambda,c}$, but by the commutator $[\tilde{H}^{\mathrm{B}}_{\Lambda,c}, \ldots]$. We will fully exploit this fact in the arguments below.

Now we start the analysis of the solutions of (5.4) for the equilibrium states $\langle \cdots \rangle_{\beta\mu}$.

Remark first that, by applying the Bogoliubov approximation the gauge invariance is broken, i.e. H^{B}_{Λ} is gauge-invariant, $\tilde{H}^{\mathrm{B}}_{\Lambda,c}$ is no longer gauge-invariant for all $c \neq 0$.

On the other hand the space translation invariance is conserved by the Bogoliubov approximation. Indeed, denote by τ_x the space translation automorphism over the distance $x \in \mathbb{R}^3$. One checks that if a^{Λ}_k stands for the annihilation operator a_k, referring to the volume Λ; then

$$\tau_x a^{\Lambda}_k = \mathrm{e}^{\mathrm{i}k \cdot x} a^{\Lambda + x}_k$$

and hence $\tau_x \tilde{H}^{\mathrm{B}}_{\Lambda,c} = \tilde{H}^{\mathrm{B}}_{\Lambda+x,c}$, expressing the translation invariance of the system. Therefore any equilibrium state, i.e. from now on any solution of the correlation inequality (5.4), is in a natural way *supposed* to be *translation-invariant*. In other words we assume that there is no spontaneous breaking of this translation symmetry. Yet we are unable to show this property, but we strongly believe that this is not really a condition. Another way of looking at this is that we *restrict* our attention to the *translation-invariant* equilibrium states.

Translation-invariant states have the property that they can be decomposed into *ergodic* or *extremal translation-invariant* states. In technical terms this means that there exists a probability measure denoted by ν, with support on the set $\mathcal{E}$ of ergodic states and such that [30, chapter 4]

$$\langle \cdots \rangle_{\beta,\mu} = \int_{\mathcal{E}} \mathrm{d}\nu(\eta) \eta(\cdots) \,.$$

By this formula and because of the convexity of both sides of the correlation inequality (5.4), it is sufficient to consider the solutions $\eta \in \mathcal{E}$ of the inequality

$$\lim_{\Lambda} \beta \eta(X^* [\tilde{H}^{\mathrm{B}}_{\Lambda,c} - \mu N_{\Lambda}, X]) \geqslant \eta(X^* X) \ln \frac{\eta(X^* X)}{\eta(X X^*)} \tag{5.6}$$

together with the equation (cf (5.5))

$$c = \lim_{\Lambda} \frac{1}{\sqrt{V}} \eta(a_0) = \lim_{\Lambda} \frac{1}{V} \int_{\Lambda} \mathrm{d}x \, \eta(a(x)) \,. \tag{5.7}$$

The main property of these ergodic states η is the existence of the ergodic means: for all local operators A, B and C the following relation holds

$$\lim_{\Lambda} \eta\left(A\left(\frac{1}{V}\int_{\Lambda} \mathrm{d}x\, \tau_x C\right)B\right) = \eta(AB)\eta(C)$$

or in a more mathematically sophisticated way it is expressed as

$$\eta - \text{weak} - \lim_{\Lambda} \frac{1}{V}\int_{\Lambda} \mathrm{d}x\, \tau_x C = \eta(C)\,.$$

In particular, take $C = a^*(y)a(y)$, $y \in \mathbb{R}^3$ where $a^{\#}(y)$ is the creation or annihilation operator at the point y. Then

$$N_\Lambda = \int_{\Lambda} \mathrm{d}x\, \tau_x(a^*(0)a(0))$$

and

$$\eta - \text{weak} - \lim_{\Lambda} \frac{N_\Lambda}{V} = \eta(a^*(0)a(0)) = \rho \tag{5.8}$$

where ρ is the particle density in the state η.

Now we come back to the correlation inequality (5.6) for the ergodic states η. As remarked above, we concentrate on the commutator with the Hamiltonian. In particular let us focus on the term $N_\Lambda(N_\Lambda - 1)$. We have

$$\frac{1}{V}[N_\Lambda(N_\Lambda - 1), X] = \frac{N_\Lambda}{V}[N_\Lambda, X] + [N_\Lambda, X]\frac{(N_\Lambda - 1)}{V}\,.$$

As X is a local observable, $[N_\Lambda, X]$ also is local. Substituting this expression in the left-hand side of (5.6), taking the limit $\Lambda \to \infty$, and using the property (5.8), one gets

$$\eta - \text{weak} - \lim_{\Lambda} \frac{1}{V}[N_\Lambda(N_\Lambda - 1), X] = 2\rho(\eta - \text{weak} - \lim_{\Lambda}[N_\Lambda, X])$$

i.e. in the commutator term in (5.6) with the Hamiltonian one can substitute $N_\Lambda(N_\Lambda - 1)/V$ by $2\rho N_\Lambda$ and obtain the equivalent system, defined by the following, so-called effective, Hamiltonian at density ρ

$$\begin{aligned} H^\rho_\Lambda(c, \mu_\Lambda) \equiv H^\rho_\Lambda - \mu_\Lambda N_\Lambda &= \sum_k [\varepsilon_k - \mu_\Lambda + \rho v(0)] a^*_k a_k \\ &\quad + \frac{1}{2}\sum_{k\neq 0} v(k)(a^*_k a^*_{-k} c^2 + \bar{c}^2 a_{-k} a_k) + |c|^2 \sum_{k\neq 0} v(k) a^*_k a_k\,. \end{aligned} \tag{5.9}$$

As the density ρ is fixed, the chemical potential μ_Λ should be determined by the equation

$$\rho = \lim_{\Lambda'} \eta\left(\frac{N_{\Lambda'}}{V'}\right) = \frac{\mathrm{Tr}\, \mathrm{e}^{-\beta H^\rho_\Lambda(c,\mu_\Lambda)} N_\Lambda / V}{\mathrm{Tr}\, \mathrm{e}^{-\beta H^\rho_\Lambda(c,\mu_\Lambda)}} \tag{5.10}$$

for all volumes Λ. This fixes μ_Λ as a function of β and ρ. Also (cf (5.7))

$$c = \lim_{\Lambda} \eta\left(\frac{a_0}{\sqrt{V}}\right) \tag{5.11}$$

and using (5.9)

$$|c|^2 = \lim_{\Lambda} \eta\left(\frac{a^*_0 a_0}{V}\right) = \lim_{\Lambda} \frac{\mathrm{Tr}\, \mathrm{e}^{-\beta H^\rho_\Lambda(c,\mu_\Lambda)} a^*_0 a_0 / V}{\mathrm{Tr}\, \mathrm{e}^{-\beta H^\rho_\Lambda(c,\mu_\Lambda)}}\,. \tag{5.12}$$

We have shown the following.

Proposition 5.2. The model (5.1) has a superstable Bogoliubov-approximated form $\tilde{H}^{B}_{\Lambda,c}$ (5.3); its equilibrium states at fixed density ρ are convex combinations of the ergodic solutions η of the correlation inequalities. For local X

$$\lim_{\Lambda} \beta\eta(X^*[H^{\rho}_{\Lambda}(c,\mu_\Lambda),X]) \geqslant \eta(X^*X)\ln\frac{\eta(X^*X)}{\eta(XX^*)} \tag{5.13}$$

with μ_Λ and c determined by (5.11) and (5.12). □

The result of all this is that our problem is reduced to the solution of the system with Hamiltonian (5.9). It is bilinear in the creation and annihilation operators. For each finite volume it can be diagonalized by a usual Bogoliubov transformation. In what follows the phase of the parameter c will not enter in an essential way. For the next analysis we take $c = \bar{c}$, and the Hamiltonian (5.9) reads as follows:

$$\begin{aligned} H^{\rho}_{\Lambda}(c,\mu_\Lambda) &= \sum_{k\neq 0}[\varepsilon_k - \mu_\Lambda + \rho v(0) + |c|^2 v(k)]a^*_k a_k \\ &\quad + \frac{|c|^2}{2}\sum_{k\neq 0} v(k)(a^*_k a^*_{-k} + a_{-k}a_k) + [v(0)\rho - \mu_\Lambda]a^*_0 a_0\,. \end{aligned} \tag{5.14}$$

After Bogoliubov transformation $a_k = u_k b_k + v_k b^*_{-k}$ (cf section 2), it takes the form

$$H^{\rho}_{\Lambda}(c,\mu_\Lambda) = \sum_{k\neq 0} E^{\rho}_k b^*_k b_k + \frac{1}{2}\sum_{k\neq 0}(E^{\rho}_k - f^{\rho}_k) + [v(0)\rho - \mu_\Lambda]a^*_0 a_0 \tag{5.15}$$

where E^{ρ}_k is called the quasi-particle spectrum given by

$$\left.\begin{aligned} E^{\rho}_k &= [(f^{\rho}_k - h^{\rho}_k)(f^{\rho}_k + h^{\rho}_k)]^{1/2} \\ f^{\rho}_k &= \varepsilon_k - \mu_\Lambda + \rho v(0) + |c|^2 v(k) \\ h^{\rho}_k &= |c|^2 v(k)\,. \end{aligned}\right\} \tag{5.16}$$

Remark that $E^{\rho}_k \geqslant 0$ only if $\mu_\Lambda \leqslant \rho v(0)$.

To analyse the Bose-condensate problem for the Hamiltonian (5.14) one has to take the limit $\Lambda \to \mathbb{R}^d$, keeping the density ρ fixed and such that (5.10)–(5.12) are satisfied.

A straightforward computation of (5.12) yields for each finite volume Λ

$$|c|^2 = \lim_{\Lambda}\frac{1}{V}\frac{1}{e^{\beta[v(0)\rho-\mu_\Lambda]}-1} \tag{5.17}$$

and of (5.10)

$$\rho = \frac{1}{V}\frac{1}{e^{\beta[v(0)\rho-\mu_\Lambda]}-1} + \frac{1}{V}\sum_{k\neq 0}\frac{1}{2}\left(\frac{f^{\rho}_k}{E^{\rho}_k}\coth\frac{\beta E^{\rho}_k}{2} - 1\right). \tag{5.18}$$

From (5.17) and (5.18) one gets

$$\rho = |c|^2 + I_d(\beta,\bar{\mu} - \rho v(0), |c|) \tag{5.19}$$

where $\bar{\mu} = \lim_{\Lambda} \mu_{\Lambda}$ (see (5.10)) and

$$I_d(\beta, \mu - \rho v(0), |c|) = \left(\frac{1}{2\pi}\right)^d \int \mathrm{d}^d k \frac{1}{2}\left(\frac{f_k^{\rho}}{E_k^{\rho}} \coth \frac{\beta E_k^{\rho}}{2} - 1\right). \tag{5.20}$$

For $\mu \leqslant \rho v(0)$ and $d > 2$, this integral is well defined and finite.

As in [6 or 30, theorem 5.2.30] define

$$\rho_c(\beta) = \sup_{\substack{x \leqslant 0 \\ y \geqslant 0}} I_d(\beta, x, y) = I_d(\beta, 0, 0) \tag{5.21}$$

and $\mu_c = \rho_c(\beta) v(0)$.

If $\rho < \rho_c(\beta)$, then $\bar{\mu}$ is the unique root of the equation

$$I_d(\beta, \bar{\mu} - \rho v(0), |c| = 0) = \rho \tag{5.22}$$

and $\lim_{\Lambda} \mu_{\Lambda} = \bar{\mu} \leqslant \mu_c(\beta)$. So $|c|^2 = 0$, as follows from (5.19).

If $\rho \geqslant \rho_c(\beta)$, then

$$\lim_{\Lambda} \mu_{\Lambda} = \rho v(0) \geqslant \mu_c(\beta) \tag{5.23}$$

and

$$|c|^2 = \rho - I_d(\beta, 0, |c|) \geqslant \rho - \rho_c(\beta) \geqslant 0\,. \tag{5.24}$$

As $\beta \to I_d(\beta, 0, 0)$ is a decreasing function, there exists a low-temperature, high-density regime in which there is Bose condensation. A macroscopic portion of particles occupy the lowest energy state $k = 0$ (see (5.12)). In particular, substituting $\lim_{\Lambda} \mu_{\Lambda} = \rho v(0) \geqslant \mu_c(\beta)$ in the quasi-particle spectrum (5.16) we get the important result

$$\lim_{\Lambda} E_k^{\rho} = \{\varepsilon_k[\varepsilon_k + 2|c|^2 v(k)]\}^{1/2}\,. \tag{5.25}$$

Hence, if $\varepsilon_k = k^2/2m$, then in the condensed phase we get for small k

$$\lim_{\Lambda} E_k^{\rho} \simeq \left(\frac{|c|^2 v(0)}{m}\right)^{1/2} |k| \tag{5.26}$$

and the 'roton' minimum around momenta k_{rot}, where $v'(k) < 0$ is minimal.

Notice the difference in the expressions (5.26) and (2.12). In fact we proved the following.

Proposition 5.3. The model (5.3), a special Bogoliubov-approximated form of (5.1), shows rigorously boson condensation in the low-temperature, high-density regime and has a linear spectrum near $k = 0$. □

This result proves that our modified Bogoliubov model explains rigorously superfluidity according to the views of Landau and Lifshitz [22]. Remark that our model becomes superfluid only in the condensed phase $|c|^2 \neq 0$ (see (5.26)) i.e. for $\rho > \rho_c(\beta)$ or $\mu > \mu_c(\beta)$ (see (5.23)–(5.25)).

Now we compare our 'minimal' extension of the Bogoliubov model (5.3) with Bogoliubov approximation (4.2) for the full Hamiltonian (5.1).

The latter one gives us instead of (5.3) the operator in $\mathcal{F}'_\Lambda$:

$$\tilde{\mathcal{H}}^{\mathrm{B}}_\Lambda(c,\mu) = \mathcal{H}^{\mathrm{B}}_\Lambda(c,\mu) + \frac{v(0)}{2V}(N'_\Lambda)^2 \tag{5.27}$$

where the first term coincides with the corresponding approximation for the Bogoliubov Hamiltonian (see (2.4), (4.2)) and $N'_\Lambda = \sum_{k\neq 0} a^*_k a_k$. Hamiltonian (5.27) commutes with the total momentum operator $P_\Lambda = \sum_{k\neq 0} k a^*_k a_k$ in the volume Λ. Therefore, as above, referring to *ergodic translation-invariant* states $\eta_{\rho'}(-)$ generated by (5.27), we can consider the effective Hamiltonian $\mathcal{H}^{\rho'}_\Lambda(c,\mu)$ corresponding to the fixed particle density ρ' above the ground state $k = 0$ (cf (5.10))

$$\rho' = \eta_{\rho'} - \text{weak} - \lim_\Lambda \langle N'_\Lambda / V\rangle_{\mathcal{H}^{\rho'}_\Lambda(c,\mu)} \tag{5.28}$$

where, according to (2.4) and (5.27), we have

$$\mathcal{H}^{\rho'}_\Lambda(c,\mu) = \mathcal{H}^{\mathrm{B}}_\Lambda(c,\mu - v(0)\rho')\,. \tag{5.29}$$

After Bogoliubov transformation (cf (2.5)–(2.9)) one gets

$$\tilde{\mathcal{H}}^{\rho'}_\Lambda(c,\mu) = \sum_{k\neq 0} E^{\rho'}_k b^*_k b_k + \frac{1}{2}\sum_{k\neq 0}(E^{\rho'}_k - f^{\rho'}_k) - V(\mu|c|^2 - \tfrac{1}{2}|c|^4 v(0)) \tag{5.30}$$

where

$$\left.\begin{aligned} f^{\rho'}_k &= \varepsilon_k - \mu + v(0)\rho' + v(0)|c|^2 + v(k)|c|^2 \\ h^{\rho'}_k &= |c|^2 v(k) \\ E^{\rho'}_k &= (f^{\rho'2}_k - h^{\rho'2}_k)^{1/2}\,. \end{aligned}\right\} \tag{5.31}$$

Comparing (5.31) with (5.16) we see that they coincide if one identifies $|c|^2 + \rho'$ with ρ, which has a clear interpretation. Remark that in contrast with (5.15), Hamiltonian (5.30) controls only density of particles in the excited states. Therefore, we cannot fix a total density of particles as we did above to examine Bose condensation—see (5.15) and (5.18). Instead one has to analyse solutions $\bar{\rho}'(\mu,c)$ of the self-consistency equation (5.28) for different μ and $|c|$ in the domain of stability of Hamiltonian (5.27), i.e. $E^{\rho'}_k \geqslant 0$ or $\mu - v(0)(\rho' + |c|^2) \leqslant 0$.

By superstability of (5.1) and, consequently, of (5.27) the chemical potential μ is allowed to run over $\mathbb{R}$. Then turning to parameter $|c|$ we have to ensure the existence of the solution of (5.28) (corresponding to the existence of the particle density above the ground state) for any fixed μ.

In the thermodynamic limit (5.28) gets the form

$$\rho' = I_d(\beta, \mu - v(0)(\rho' + |c|^2), |c|) \tag{5.32}$$

where $I_d(\beta, x, y)$ is the integral (5.20). This integral is a decreasing function of the variables $x \in \mathbb{R}_-$ and $y \in \mathbb{R}_+$ attaining its sup at $x = y = 0$. According to (5.32) this corresponds to the maximal density ρ', i.e. $\rho'_{\max} = I_d(\beta, 0, 0) = \rho_c$ (cf (5.21)). Simultaneously, condition $x = 0$ defines the critical value of the chemical potential

$$\mu_c = v(0)\rho_c = v(0)I_d(\beta, 0, 0)\,. \tag{5.33}$$

If μ becomes larger than μ_c, then (5.32) can be satisfied only if $|c| \neq 0$. This is to compensate the increment of μ in the second argument of I_d (note that $\bar{\rho}'$ is already at its maximum), guaranteeing the stability condition. Then, as in the case of our model (5.3), the condensate density $|\tilde{c}|^2 \neq 0$ for $\mu > \mu_c$ and it satisfies the following equations:

$$\left.\begin{aligned} &\mu - v(0)(|\tilde{c}|^2 + \bar{\rho}'(\mu, \tilde{c})) = 0 \\ &\bar{\rho}'(\mu, \tilde{c}) = I_d(\beta, 0, |\tilde{c}|) \end{aligned}\right\} \tag{5.34}$$

(cf (5.23), (5.24)).

Hence, the Bogoliubov approximation for the model (5.1) gives the same results for ρ_c, μ_c and the same equations for $\bar{\rho}'$ and $\tilde{c}$ as we got above in our approach (5.3).

It is worth remarking that one could also treat the Hamiltonian (5.27), fixing the parameter c, following the ansatz of section 4. Namely, referring to the inequality

$$p_\Lambda[\mathcal{H}_\Lambda^{\rho'}(c, \mu)] \leqslant p_\Lambda[H_\Lambda^{\mathrm{B}} - \mu N_\Lambda + v(0)\rho' N_\Lambda] \tag{5.35}$$

(cf (4.6)) where ρ' satisfies (5.28), we can define $\tilde{c}$ by the condition

$$\sup_c p_\Lambda[\mathcal{H}_\Lambda^{\rho'}(c, \mu)] = p_\Lambda[\mathcal{H}_\Lambda^{\rho'}(\tilde{c}, \mu)]\,. \tag{5.36}$$

Remark that for $\mu < \mu_c$ and the parameter c in the domain of stability $\{c : \mu - v(0)(\rho' + |c|^2) \leqslant 0\}$ the solution of (5.28) satisfies the inequality $\bar{\rho}'(\mu, c) > \mu/v(0)$. Therefore, by the proposition 4.4 the sup in (5.36) is attained at $\tilde{c} = 0$. On the other hand, for $\mu \geqslant \mu_c$ the sup in (5.36) is attained on the boundary of the stability domain, corresponding to the first of the equations (5.34) constrained by the second equation for $\bar{\rho}'$. So, we obtain the same result as above in (5.23)–(5.24).

6. Concluding remarks

We proved that the conventional Bogoliubov Hamiltonian (2.3) is thermodynamically unstable for $\mu > 0$ (section 4), while for $\mu \leqslant 0$ it seems to be equivalent to the ideal Bose gas. The latter statement is proved only in the domain (3.14).

We also show that the extension of the Hamiltonian (2.3), obtained by including the full forward scattering term (N^2-term), yields a new model (5.1) which is superstable. It is rigorously solved in its presentation (5.3). Considering ergodic translation-invariant equilibrium states yields rigorous arguments for deriving the 'superfluid' spectrum of the collective excitations, i.e. a gapless and linear spectrum for small wavevectors. This results in remedies to the conventional sloppy arguments in doing the Bogoliubov transformation for the Hamiltonian with the N^2-term. We prove that the Hamiltonian (5.1) in its formulation (5.3) has a 'superfluid' spectrum only in the condensed phase. This phase corresponds to Bose condensation occuring at high densities ($\rho > \rho_c$) or large chemical potentials ($\mu > \mu_c = v(0)\rho_c$) and, when it is satisfied, to the equality $\mu = \rho v(0)$. This relation has been predicted on the basis of lowest-order perturbation theory and put in by hand in the conventional theory of superfluidity [20, 21], together with the assumption that condensate density equals the total density. Our critical density ρ_c coincides with the one of the ideal Bose gas. An analogous result is known for the imperfect Bose gas [8–10].

For $\rho \leqslant \rho_c$ ($\mu \leqslant \mu_c$) the condensate density vanishes, and the pressure and the expression for the total particle density correspond with the equivalent one for the imperfect Bose gas [8–10]. But for $\rho > \rho_c$ the thermodynamic behaviour of our model is intrinsically different from the imperfect Bose gas in many respects. The main difference can be seen as the outcome of the inequality

$$\bar{\rho}'(\beta,\mu) = I_d(\beta, 0, |c| \neq 0) < I_d(\beta, 0, 0) \equiv \rho_c\,.$$

It implies that the density of particles above the ground state decreases (for $\mu > \mu_c$) if the density of condensed particles $|c|^2$ increases. This is in contrast with the imperfect (as well as the ideal) Bose gas for which $\bar{\rho}'(\beta,\mu)$ saturates at ρ_c.

Finally we remark that all our proofs are rigorous, except for the fact that we did not enter into the peculiarities and details of the thermodynamic limit as such. We have assumed always that the limit $\Lambda \to \mathbb{Z}^d$ exists. We leave this point to another occasion.

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